



Berry, M. V. (2019). Minimal model for tidal bore revisited. *New Journal of Physics*, 21, [073021]. <https://doi.org/10.1088/1367-2630/ab2b19>

Publisher's PDF, also known as Version of record

License (if available):
CC BY

Link to published version (if available):
[10.1088/1367-2630/ab2b19](https://doi.org/10.1088/1367-2630/ab2b19)

[Link to publication record in Explore Bristol Research](#)
PDF-document

This is the final published version of the article (version of record). It first appeared online via IOP Publishing at <https://iopscience.iop.org/article/10.1088/1367-2630/ab2b19> . Please refer to any applicable terms of use of the publisher.

University of Bristol - Explore Bristol Research

General rights

This document is made available in accordance with publisher policies. Please cite only the published version using the reference above. Full terms of use are available:
<http://www.bristol.ac.uk/red/research-policy/pure/user-guides/ebr-terms/>



PAPER • OPEN ACCESS

Minimal model for tidal bore revisited

To cite this article: M V Berry 2019 *New J. Phys.* **21** 073021

Recent citations

- [Analytic iteration procedure for solitons and traveling wavefronts with sources](#)
Jonas Berx and Joseph O Indekeu

View the [article online](#) for updates and enhancements.



PAPER

Minimal model for tidal bore revisited

M V Berry 

H H Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, United Kingdom

E-mail: asymptotico@bristol.ac.uk**Keywords:** wave, Hamiltonian, asymptotic, caustic, nonlinearity, Airy function

OPEN ACCESS

RECEIVED
28 April 2019**REVISED**
1 June 2019**ACCEPTED FOR PUBLICATION**
19 June 2019**PUBLISHED**
8 July 2019Original content from this work may be used under the terms of the [Creative Commons Attribution 3.0 licence](https://creativecommons.org/licenses/by/4.0/).

Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

**Abstract**

This develops a recent analysis of gentle undular tidal bores (2018 *New J. Phys.* **20** 053066) and corrects an error. The simplest linear-wave superposition, of monochromatic waves propagating according to the shallow-water dispersion relation, leads to a family of profiles satisfying natural tidal bore boundary conditions, involving initial smoothed steps with different shapes. These profiles can be uniformly approximated to high accuracy in terms of the integral of an Airy function with deformed argument. For the long times corresponding to realistic bores, the profiles condense asymptotically onto the previously obtained integral-Airy function with linear argument: as the bore propagates, it forgets the shape of the initial step. The integral-Airy profile expands slowly, as the cube root of time, rather than advancing rigidly. This ‘minimal model’ leads to simple formulas for the main properties of the profile: amplitude, maximum slope, ‘wavelength’, and steepness; and an assumption about energy loss suggests how the bore weakens as it propagates.

1. Introduction

In some of the world’s rivers open to an ocean, the advancing tide develops into a smooth front followed by a train of waves with a characteristic shape, travelling far upstream: an undular bore. In a recent paper [1], I presented a ‘minimal model’ [2] in which the bore profile is the integral of an Airy function. That theory predicted, erroneously, that the profile would propagate rigidly, that is, without change of scale. My purpose here is to present the theory *ab initio*, and extend it, avoiding the error. The simplest model now predicts the same integral-Airy profile, but with a nonlinearly stretched argument. For long times, this reduces to the earlier integral-airy profile, i.e. without the nonlinear stretching, but instead of propagating rigidly the profile slowly expands as the bore advances.

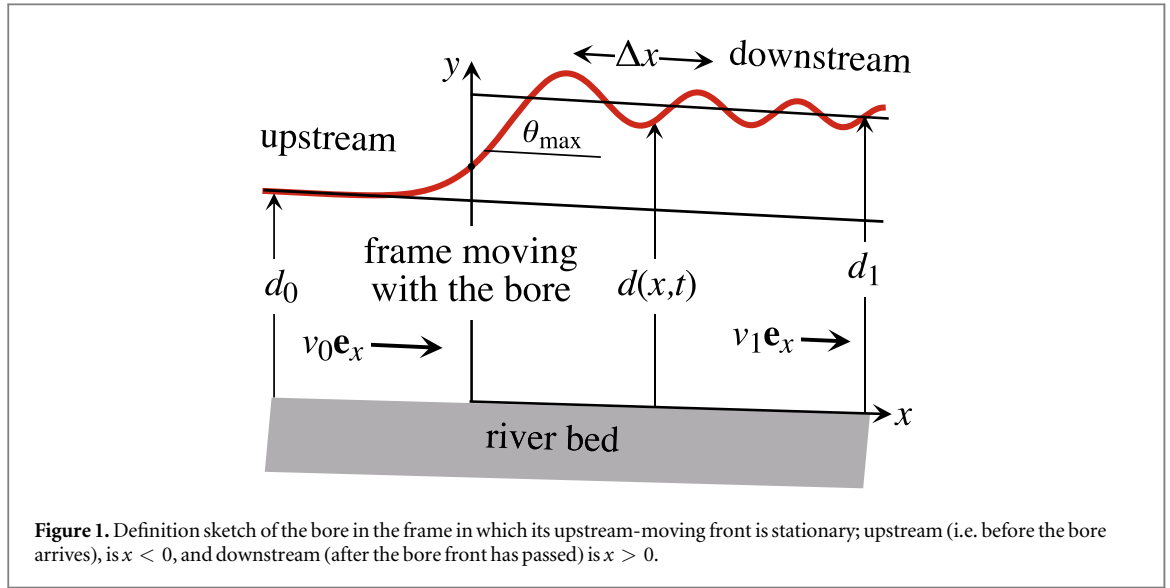
There is a substantial literature on tidal bores [3–9], in which the emphasis is on effects associated with nonlinearity, often modelled by the Korteweg–de Vries (KdV) equation [10–18]. The distinctive feature of the present version of the minimal model is the systematic exploration of the description based solely on linear waves. The outcome will be a family of bore shapes (section 2), uniformly approximated to high accuracy (section 3), with the expanding integral-Airy profile (section 4) emerging as a realistic approximation for natural bores, stable in the sense that the details of the initial step get forgotten as the bores propagate. The theory makes a number of quantitative predictions, listed in the concluding section 5.

Referring to figure 1, schematically depicting the bore in a frame moving upstream with it, the aim is to calculate the water depth $d(x, t)$ between its value d_0 in the downflowing river and its value $d_1 (>d_0)$ in the advancing tide. For the gentle bores described by linear-wave theory, the important parameter

$$r \equiv \frac{d_1}{d_0} \quad (1.1)$$

is only slightly greater than unity.

The starting-point is the dispersion relation relating the frequency ω and wavenumber k for monochromatic waves on water of depth d :



$$\omega_0(k) = \pm \sqrt{gk \tanh dk}. \quad (1.2)$$

This describes waves on water that is not flowing, but in the frame moving with the bore, water flows downstream, i.e. towards positive x . The bore profile will be built as a superposition of monochromatic waves. For downflowing speed v , transformation to the bore frame gives, for waves that advance upstream in the land frame, the monochromatic wave and the dispersion relation (Hamiltonian)

$$\exp(i(kx - \omega(k)t)), \quad \omega(k) = vk - \sqrt{gk \tanh dk}. \quad (1.3)$$

What is v ? Standard hydraulic jump theory [7], in which incompressibility of water is combined with Newtonian dynamics ('momentum equation') relates the downflowing upstream and downstream speeds (in the bore frame) v_0 and v_1 to the corresponding depths d_0 and d_1 :

$$v_0 = \sqrt{\frac{1}{2}gd_1(1+r)}, \quad v_1 = \sqrt{\frac{1}{2}gd_0(1/r+1)} = \frac{v_0}{r}. \quad (1.4)$$

For r only slightly greater than unity, v_0 (supercritical) is only slightly greater than v_1 (subcritical), and both values are close to

$$v = \sqrt{gd_0}, \quad (1.5)$$

which is the value that will be used in the dispersion relation (1.3). The supercritical flow speed upstream and the subcritical speed downstream formed the basis of the analogy with horizons in relativity physics, described in [1] (the new feature described here, that the bore expands as it propagates, seems to have no counterpart for relativistic horizons).

2. Family of bore profiles

It is convenient to transform to the following dimensionless variables:

$$X \equiv \frac{x}{d_0}, \quad K \equiv kd_0, \quad T \equiv t \sqrt{\frac{g}{d_0}} = \frac{t}{d_0} \sqrt{gd_0} = \frac{vt}{d_0}. \quad (2.1)$$

The last equality shows that T is the distance travelled by the bore since its nominal birth at $T = 0$, divided by the depth. For all realistic bores, T is very large, a fact we will exploit later (for a bore that has travelled 10 km on a river of depth 2 m, $T = 5000$).

We will create wave superpositions that describe the bore profile above the downflowing depth d_0 , thus:

$$d(x, t) = d_0 + (d_1 - d_0)\eta(X, T), \quad \eta(-\infty, T) = 0, \quad \eta(+\infty, T) = 1. \quad (2.2)$$

(The error in [1] was to employ linear waves to represent the entire depth $d(x)$, rather than the difference $d(x) - d_0$.) Using (1.3) and (2.1), the superposition can be written as

$$\eta(X, T) = \int_C dK g(K) \exp(i(KX - (K - \sqrt{K \tanh K})T)), \quad (2.3)$$

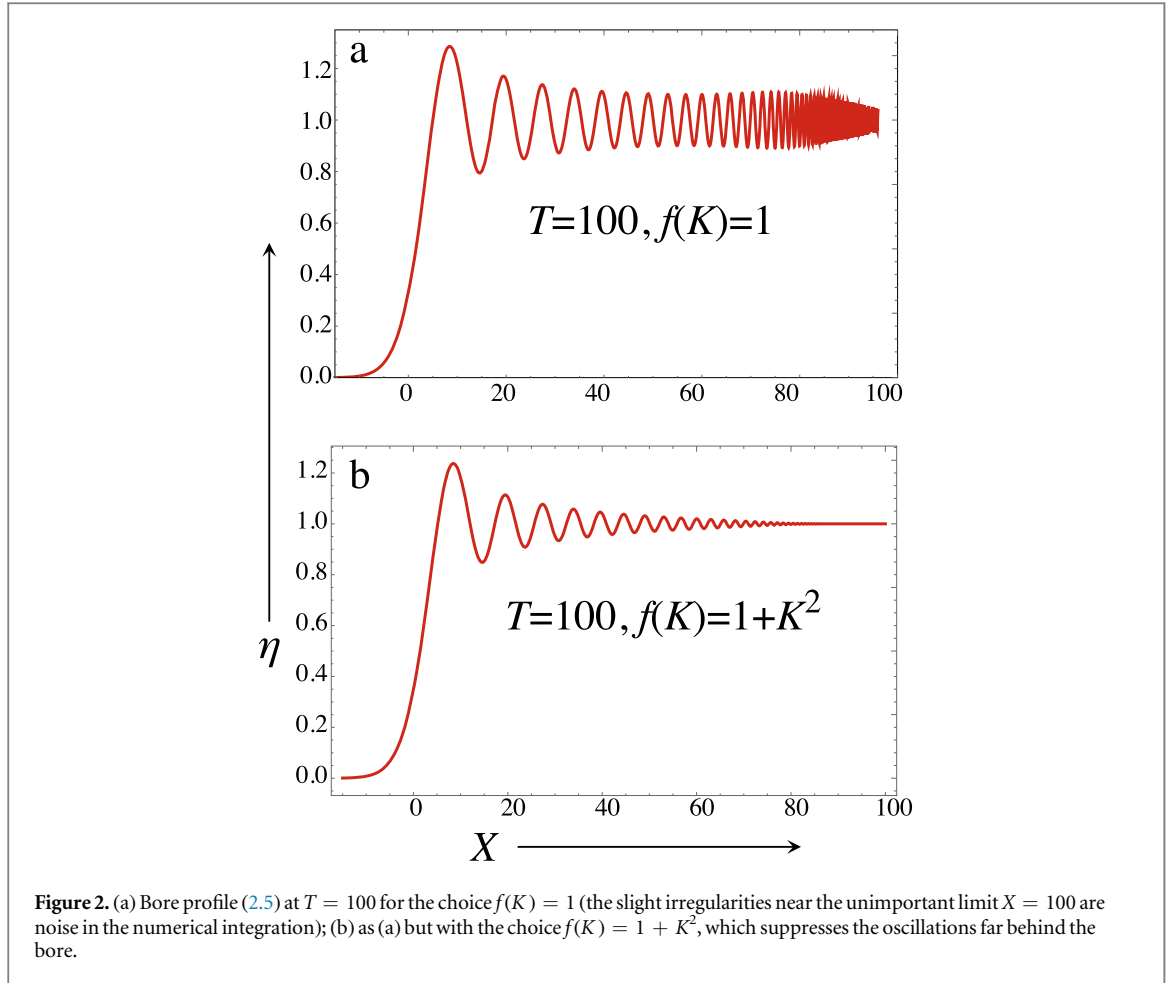


Figure 2. (a) Bore profile (2.5) at $T = 100$ for the choice $f(K) = 1$ (the slight irregularities near the unimportant limit $X = 100$ are noise in the numerical integration); (b) as (a) but with the choice $f(K) = 1 + K^2$, which suppresses the oscillations far behind the bore.

with amplitude $g(K)$ and contour C to be determined. This type of superposition has been systematically explored for tsunami propagation [19]. The difference here is the need to accommodate the transition from $\eta = 0$ to $\eta = 1$ between $T = -\infty$ and $T = +\infty$; it means that $g(K)$ must have a simple pole at $K = 0$, with residue unity, that is

$$g(K) = \frac{1}{2\pi i K f(K)}, \quad f(0) = 1, \quad (2.4)$$

and C must pass below the real axis. This leads to the following family of bore profiles:

$$\eta(X, T) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{dK}{K f(K)} \sin(KX - (K - \sqrt{K \tanh K})T). \quad (2.5)$$

In this theory, $T = 0$ represents the birth of the bore, when the profile is a simple jump, with no waves. The shape of this initial jump is represented by the function $f(K)$. For the simplest choice $f(K) = 1$, the jump is a sharp step; if $f(K)$ is an increasing function, the jump is smooth. Later (section 4) we will find that the evolution after long times is independent of $f(K)$.

Figure 2(a) shows the profile for $f(K) = 1$, for $T = 100$. The oscillations get faster as X increases, stopping abruptly near $X = T$. These very fast oscillations are unphysical because $X = T$ corresponds to following the bore all the way down-river to the place where it was born, where numerous factors are likely to spoil the idealisation in this minimal model. These far-downstream oscillations can be suppressed by choosing $f(K)$ as an increasing function of K , as figure 2(b) illustrates for $f(K) = 1 + K^2$.

The oscillations can be understood by approximating the integral in (2.5) asymptotically for $X \gg 0$. There is a pole, contributing $1/2$, and a stationary-phase point (saddle) that describes the oscillations. The stationary-phase point is at K_c , involving the group velocity V_g and determined by

$$\frac{X}{T} \equiv u = 1 - \partial_K \sqrt{K \tanh K} \equiv 1 - V_g(K) \Rightarrow K = K_c(u). \quad (2.6)$$

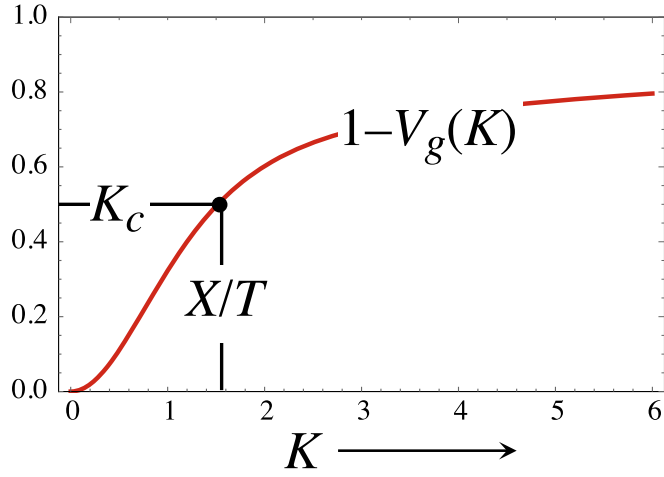


Figure 3. Construction of the stationary-phase point of the integrand in (2.5).

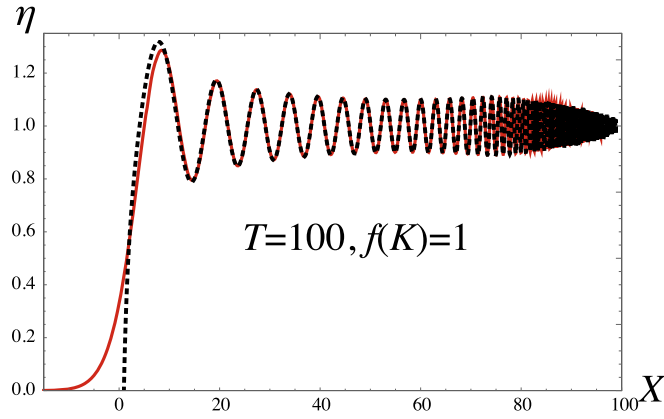


Figure 4. Full red curve: the exact profile (2.5) for $T = 100$ and $f(K)=1$ (as in figure 2); dotted black curve: the stationary-phase approximation (2.7).

This exemplifies a familiar wave phenomenon: the wavenumber of the local oscillations near X, T is that of the monochromatic wave that travels to X in time T with the group velocity (in this case, after transformation to the bore frame). Figure 3 illustrates this construction.

The standard stationary-phase method [20] now leads to the approximate profile

$$\eta(X, T) \approx \eta_{\text{asyp}}(X, T) = 1 - \sqrt{\frac{2}{\pi T V_g'(K)}} \frac{\cos\left(K(X - T) + T\sqrt{K \tanh K} + \frac{1}{4}\pi\right)}{K f(K)} \Bigg|_{K=K_c(X/T)}. \quad (2.7)$$

As figure 4 illustrates, this accurately describes the oscillations, even for rather small X , but of course it fails near the front, i.e. $X = 0$, and upstream, i.e. $X < 0$.

3. Uniform approximation for profile

We now obtain an approximation to the exact linear-wave profile (2.5) that incorporates both the large X approximation (2.7) and the behaviour near the front and upstream, that is, a large T approximation uniformly valid for all X . Reverting to the contour integral (2.3) and (2.4), we eliminate the pole by introducing a further integral:

$$\eta(X, T) = T \int_{-\infty}^{X/T} du H(u; T), \quad (3.1)$$

where

$$H(u; T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dK}{f(K)} \exp(iT(-Ku + (K - \sqrt{K \tanh K}))). \quad (3.2)$$

To simplify this integral, we apply the now-standard procedure [21–23] of transforming the integration variable K to a new variable Q , replacing the exponent by a simpler function with the same qualitative behaviour. In this case the behaviour is linear + cubic, so we choose the transformation

$$T(-Ku + (K - \sqrt{K \tanh K})) \equiv -Q\xi(u, T) + \frac{1}{3}Q^3. \quad (3.3)$$

The new function $\xi(u, t)$ must be chosen to make the transformation smooth and reversible, so the stationary point $K = K_c(u)$ of the left side, defined by (2.6), must correspond to the stationary point $Q = \sqrt{\xi}$ of the right side. This leads to

$$\xi(u, T) = \left[\frac{3}{2}T(-Ku + (K - \sqrt{K \tanh K})) \right]_{K=K_c(u)}^{2/3}. \quad (3.4)$$

(The same transformation has been used to simplify tsunami propagation [19].) For $u < 0$, i.e. $X < 0$ (upstream of the bore), the correct branch is the one for which ξ is negative real. Thus

$$H(u; T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dQ}{f(K(Q))} \frac{dK(Q)}{dQ} \exp\left(i\left(-Q\xi(u, T) + \frac{1}{3}Q^3\right)\right). \quad (3.5)$$

This is still exact. The technique of uniform approximation, whose lowest order (in T) will suffice here, consists in replacing $K(Q)$ in the prefactor by the stationary point $K_c(u)$. The derivative is determined by differentiating the transformation (3.3) twice:

$$\left(\frac{dK}{dQ}\right)_{Q=\sqrt{\xi}} = \frac{\xi^{1/4}}{\sqrt{-\frac{1}{2}V'_g(K_c(u))}}. \quad (3.6)$$

For the correct branch, this is positive real for all u . With the prefactor thus extracted, the integral is an Airy function, leading to the uniform approximation for the profile (3.1):

$$\eta(X, T) \approx \eta_{\text{uniform}}(X, T) = T \int_{-\infty}^{X/T} du \frac{\xi(u, T)^{1/4} \text{Ai}(-\xi(u, T))}{f(K_c(u)) \sqrt{-\frac{1}{2}V'_g(K_c(u))}}. \quad (3.7)$$

Since the bore front can be regarded as a caustic [1], where two ‘water rays’ ($K_c(u)$ in (2.6) and its negative) coalesce, the appearance of the Airy function, common to two-ray caustics [24] is unsurprising.

The extraordinary accuracy of this large T approximation over the whole range of X is illustrated in figure 5, even for $T = 20$ which is far smaller than any value relevant to the modelling of natural bores. There are small discrepancies; some are barely visible in figure 5(b).

4. Asymptotic emergence of expanding integral-Airy profile

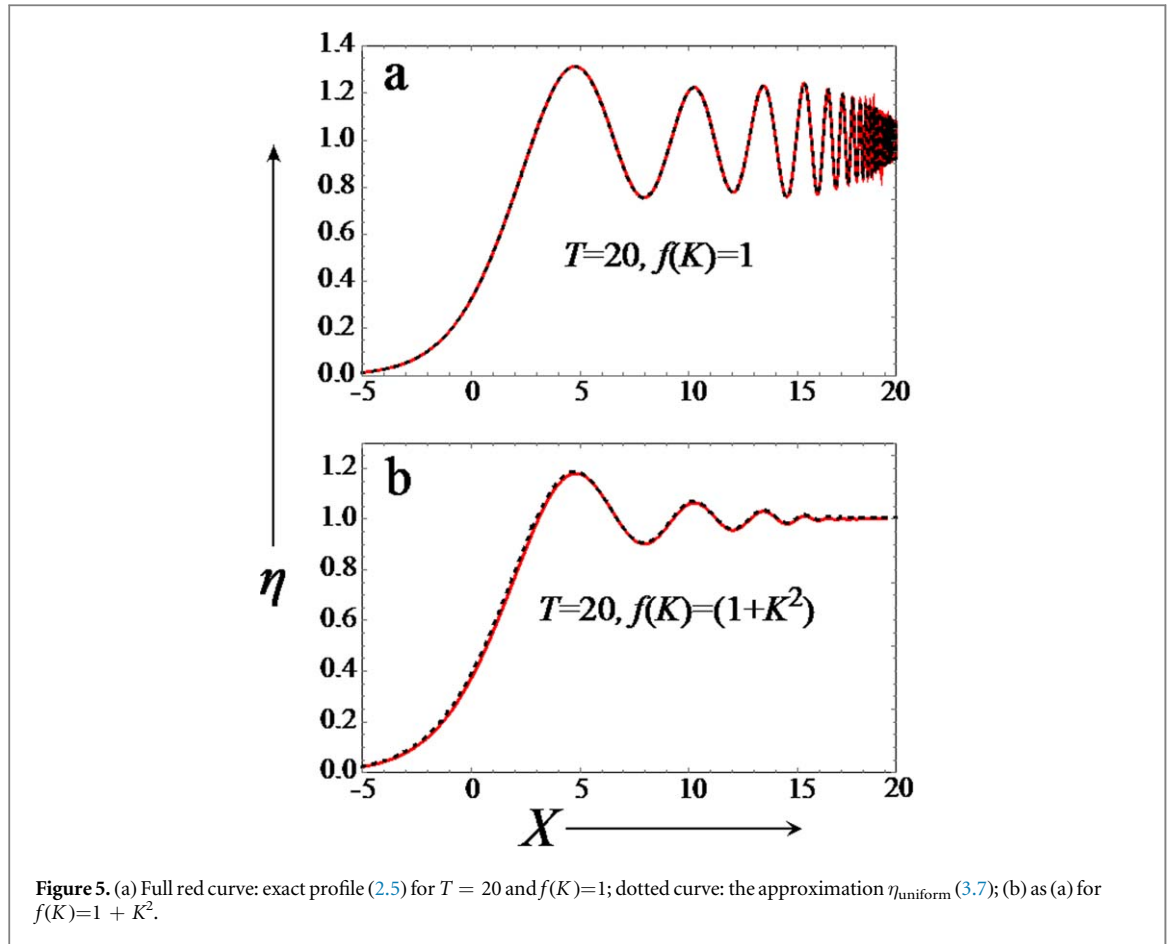
For the physical situation $T \gg 1$, with X fixed, a simplified version of the uniform approximation is valid. This could be derived by approximating (3.7), but it is simpler to exploit the fact that the oscillations are determined by the small K region in the integrand of (2.5). Then it is natural to approximate the dispersion relation in (1.3) (in the scaled variables) by

$$K - \sqrt{K \tanh K} = \frac{1}{6}K^3 + O(K^5), \quad (4.1)$$

and replace $f(K)$ with $f(0) = 1$. Now the integral can be evaluated analytically, including the region near $X = 0$ where the pole and stationary-phase point merge, with the result

$$\eta(X, T) \approx \eta_{\text{Airy}}(X, T) = \int_{-X(2/T)^{1/3}}^{\infty} du \text{Ai}(u). \quad (4.2)$$

This is our main result: the same integral-Airy profile shape obtained in [1], but now slowly expanding as the bore travels upstream.



In water-wave theory, the nonlinear dynamics with the dispersion approximated as cubic can be described [12] by the KdV equation, which in the dimensionless variables (2.1), using (2.2), and with the sign chosen to represent waves travelling upstream (negative x), is

$$\eta_T + (r - 1)\eta\eta_X - \frac{1}{6}\eta_{XXX} = 0. \quad (4.3)$$

The expanding Airy-integral profile emerges when the nonlinear term can be neglected, i.e. $r-1 \ll 1$; this solution is known [13, 25], as is its (nonuniform) asymptotics [26], but its application as a minimal model for tidal bores does not seem to have been systematically explored. The $T^{1/3}$ expansion of an Airy profile (not integrated) is familiar in the theory of tsunami propagation [27].

In the language of asymptotics, the expanding integral-Airy profile (4.2) is a transitional approximation; the range of X for which it is applicable—near the front—gets larger as T increases. Figure 6 illustrates this; for the realistic value $T = 5000$, the agreement over the range depicted is almost perfect. This indicates that the expanding Airy-integral profile is the definitive linear-wave representation of the bore profile. For later reference, figure 7 shows detail of this profile near the bore front.

The emergence of the integral-Airy profile can be illustrated explicitly for the initial condition

$$\eta(X, 0) = \frac{1}{2}(1 + \text{erf}(X/2L)), \quad (4.4)$$

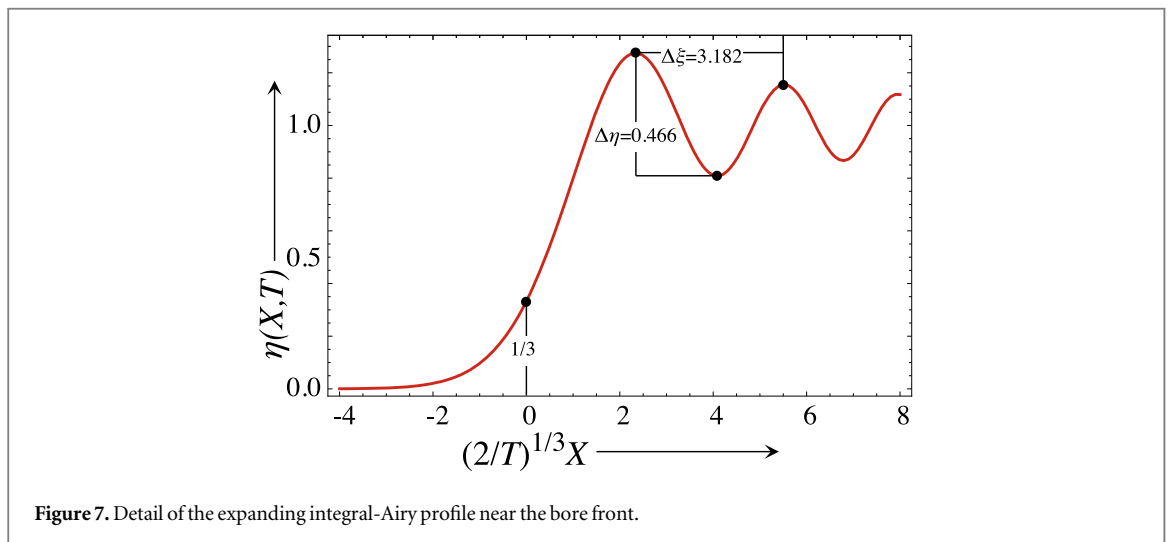
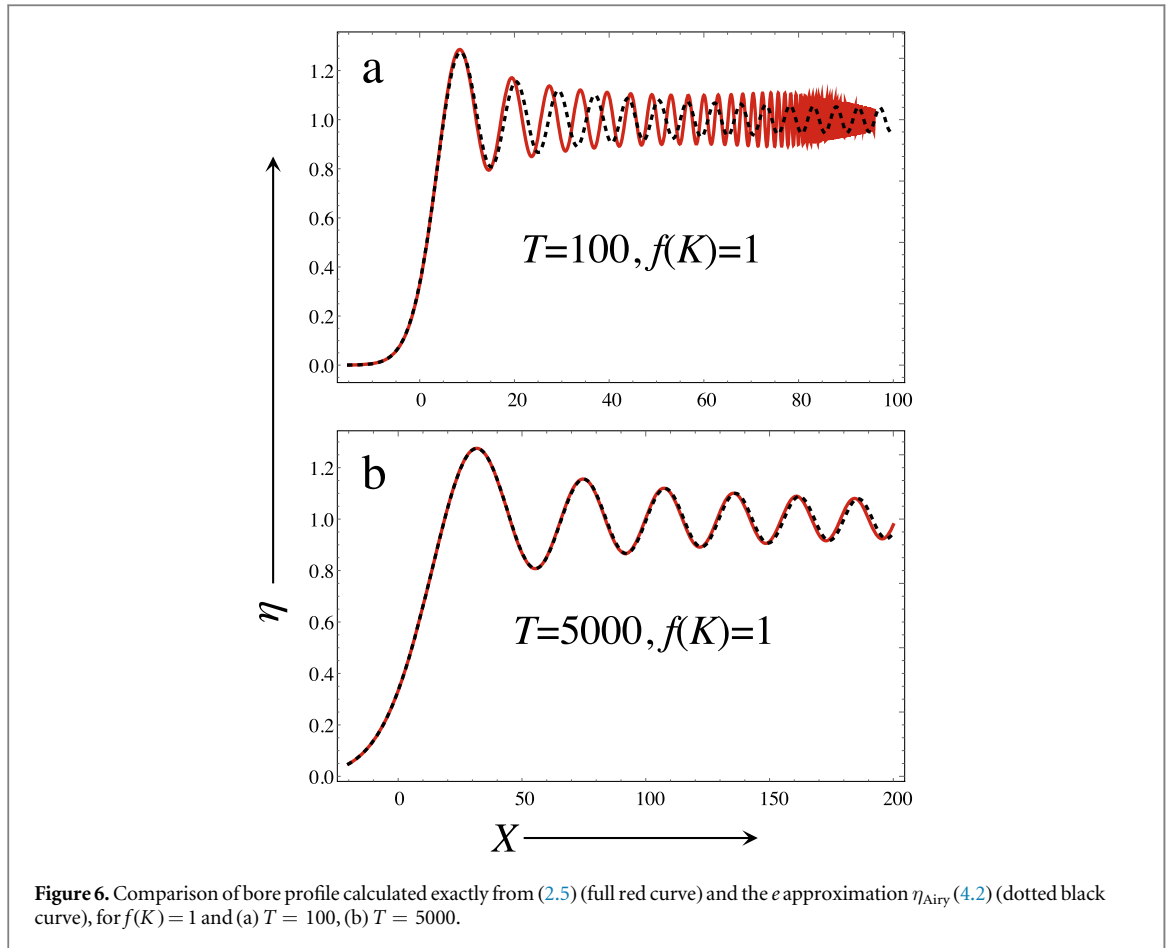
corresponding to

$$f(K) = \exp(K^2 L^2) \quad (4.5)$$

in (2.4). For this case, the exact solution of the linearised KdV equation is

$$\eta(X, T) = \exp\left(\frac{8L^6}{3T^2}\right) \int_{-X(2/T)^{1/3}}^{\infty} du \exp\left(uL^2\left(\frac{2}{T}\right)^{2/3}\right) \text{Ai}\left(u + \left(\frac{2}{T}\right)^{4/3} L^4\right). \quad (4.6)$$

It is clear that when $T \gg L^3$ this reduces to (4.2), whatever the value of the initial smoothing L : the wave forgets the shape of its initial step.



5. Consequences

The expanding integral-Airy profile (4.2) is surprisingly rich in predicting features of gentle tidal bores. Here is a list, extending and correcting section 4 of [1].

(a) A consequence of dispersion is that the bore profile (2.2) expands as it propagates. The expansion is slow: as $T^{1/3}$. For example, the Severn bore takes about $T_1 = 1.75$ h to travel from Awre, where it starts, to Stonebench, a popular viewing location (a distance of 24.1 km), and $T_2 = 2.17$ h to travel from Awre to Maisemore, where a weir destroys it (a distance of 31.8 km); therefore the expansion ratio is only about $(T_2/T_1)^{1/3} = 1.07$, which would be hard to see amid the many complicating factors associated with real bores;

I know of no relevant observations. A factor that would tend to inhibit the expansion is nonlinearity, for example as embodied in the term omitted in the KdV equation (4.3). It is hard to see how this would be relevant for gentle bores, but nonlinearity probably explains why ripples do not expand in the analogous stationary hydraulic jump in a sink [28], generated by water flowing outwards while replenished from a tap above; the depth contrast seems substantial, so this analogue bore is probably not gentle.

(b) The depth at the front of the bore, defined as $X = 0$, follows from the integral of $\text{Ai}(u)$ over $0 < u < \infty$:

$$d(0, T) = \frac{1}{3}d_0(2 + r). \quad (5.1)$$

Thus at $X = 0$ the water surface has risen by $1/3$ of its total rise between d_0 and d_1 .

(c) The slope at $X = 0$ is

$$\tan \theta(T) = (r - 1)(2/T)^{1/3} \text{Ai}(0) = 0.4473 \frac{(r - 1)}{T^{1/3}}. \quad (5.2)$$

(The maximum slope θ_{\max} , at the inflection of the profile (figure 7), is greater by a factor 1.509 (see e.g. equation (4.4) of [1]).) Thus the slope decreases as the bore propagates, assuming the depth ratio r remains constant (for a conjecture about this, see (g) below).

(d) The ‘wavelength’ λ near the front, defined as the distance between the first two wave crests, is (see figure 7)

$$\lambda(T) = d_0 \Delta \xi \frac{(r - 1)}{\tan \theta(T)} \text{Ai}(0) = d_0 \Delta \xi (T/2)^{1/3} = 2.526 d_0 T^{1/3}. \quad (5.3)$$

The increasing wavelength, as well as the decreasing slope (5.2), predicts that the bore gets gentler as it advances, reinforcing the linear-wave approximation.

(e) The ‘amplitude’ a , defined as the vertical distance between the first maximum and the first minimum, is

$$a = d_0(r - 1) \Delta \eta = 0.466 d_0(r - 1). \quad (5.4)$$

Intriguingly, a different argument, for the amplitude of waves in a monochromatic train [10], gives the same functional dependence and the slightly different constant $1/\sqrt{3} = 0.577$; and our earlier argument (involving the erroneous assumption), gave the same result (equation 4.9) of [1], after a factor $1/2$ between the two definitions is included.

(f) The ‘steepness’, defined [7] as the ratio of a and $\lambda/2$, is

$$\begin{aligned} S(T) &= \frac{a}{\frac{1}{2}\lambda(T)} = \frac{d_0(r - 1) \Delta \eta}{\frac{1}{2}\lambda(T)} = \frac{2 \tan \theta(T) \Delta \eta}{\text{Ai}(0) \Delta \xi} \\ &= 2(r - 1)(2/T)^{1/3} \frac{\Delta \eta}{\Delta \xi} = 0.369(r - 1) T^{1/3}. \end{aligned} \quad (5.5)$$

(g) Finally, it is interesting to explore implications of the physical assumption [10] that the energy known [29] to be lost in a hydraulic jump is carried away by the accompanying waves. Of course this neglects energy losses from friction (and possible gain from wind). For density ρ , the energy balance (per unit span of the bore) for monochromatic waves of wavenumber K [30] is

$$\frac{1}{4} \rho g (r - 1)^3 v d_0^2 = \frac{1}{4} \rho g a^2 v \left(1 - \frac{2K}{\sinh 2K} \right). \quad (5.6)$$

For the bore we are considering, which is not a monochromatic wave, it seems reasonable to choose $K = 2\pi d_0/\lambda \ll 1$, from which

$$K = \frac{2\pi d_0}{\lambda} \ll 1 \Rightarrow \frac{\lambda}{d_0} = \frac{2^{3/2} \pi \Delta \eta}{\sqrt{3(r - 1)}} = \frac{2.391}{\sqrt{r - 1}}, \quad (5.7)$$

extending (5.3). The analogous result in equation (4.8) of [1], although obtained using an erroneous assumption, is similar: the same dependence on r (for r near 1), and the similar constant 1.780. And the analogous theory [10] for monochromatic waves also has the same r dependence, with the not too different constant $\pi(2/3)\sqrt{2} = 2.96$. Combining (5.7) with (5.3) gives

$$r - 1 = \frac{8\pi^2 \Delta \eta^2}{3 \Delta \xi^2} \left(\frac{2}{T} \right)^{2/3} = \frac{0.896}{T^{2/3}}, \quad (5.8)$$

predicting, on this assumption about the mechanism of energy loss, that the depth ratio r gets slowly closer to 1 as the bore propagate, i.e. the bore weakens.

Acknowledgments

I thank Dr Karima Khusnutdinova, who guided me through the literature on nonlinear aspects of tidal bores, showed me the expanding integral-Airy solution of the linearised KdV equation, and insisted that the profile must expand as the bore propagates. I also thank Professor John Hannay for many helpful conversations.

ORCID iDs

M V Berry  <https://orcid.org/0000-0001-7921-2468>

References

- [1] Berry M V 2018 Minimal analytical model for undular tidal bore profile: quantum and Hawking effect analogies *New J. Phys.* **20** 053066
- [2] Batterman R W and Rice C C 2014 Minimal model explanations *Phil. Sci.* **81** 349–76
- [3] Tricker R A R 1964 *Bores, Breakers, Waves and Wakes: An Introduction to the Study of Waves on Water* (London: Mills & Boon)
- [4] Lynch D K 1982 Tidal bores *Sci. Am.* **247** 146–56
- [5] Colas A 2014 *Mascaret: l'onde Lunaire* (Anglet: Editions YEP)
- [6] Colas A 2017 *Mascaret: Prodige de la Marée* (Anglet: Editions YEP)
- [7] Chanson H 2012 *Tidal Bores, Aegir, Eagre, Mascaret, Pororoca; Theory and Observations* (Singapore: World Scientific)
- [8] Bonneton P, Bonneton N, Parisot J-P and Castelle B 2015 Tidal bore dynamics in funnel-shaped estuaries *J. Geophys. Res.: Oceans* **120** 1–19
- [9] Berchet A, Simon B, Beaudouin A, Lubin P, Rousseaux G and Huberson S 2018 Flow fields and particle trajectories beneath a tidal bore: a numerical study *Int. J. Sediment Res.* **33** 351–70
- [10] Benjamin T B and Lighthill M J 1954 On cnoidal waves and bores *Proc. R. Soc. A* **224** 448–60
- [11] Peregrine D H 1966 Calculations of the development of an undular bore *J. Fluid Mech.* **25** 321–30
- [12] Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Wiley)
- [13] Gurevich A V and Pitaevskii L P 1974 Nonstationary structure of a collisionless shock wave *Sov. Phys.—JETP* **38** 291–7
- [14] Caputo J-G and Stepanyants Y A 2003 Bore formation, evolution and disintegration into solitons in shallow inhomogeneous channels *Nonlinear Process. Geophys.* **10** 407–24
- [15] Fornberg B and Whitham G B 1978 A numerical and theoretical study of certain nonlinear wave phenomena *Phil. Trans. R. Soc. A* **289** 373–404
- [16] Kamchatnov A M, Kuo Y-H, Lin T-C, Horng T-L, Guo S-C, Clift R, El G A and Grimshaw R H J 2012 Undular bore theory for the Gardner equation *Phys. Rev. E* **86** 036605
- [17] Alfatih A and Kalisch H 2012 A dispersive model for undular bores *Anal. Math. Phys.* **2** 347–66
- [18] An X, Marchant T R and Smyth N F 2018 Dispersive shock waves governed by the Whitham equation and their stability *Proc. R. Soc. A* **474** 20180278
- [19] Berry M V 2005 Tsunami asymptotics *New J. Phys.* **7** 129
- [20] Wong R 1989 *Asymptotic Approximations to Integrals* (New York: Academic)
- [21] Chester C, Friedman B and Ursell F 1957 An extension of the method of steepest descents *Proc. Camb. Phil. Soc.* **53** 599–611
- [22] Temme N M 2015 *Asymptotic Methods for Integrals* (Singapore: World Scientific)
- [23] DLMF 2010 *NIST Handbook of Mathematical Functions* (Cambridge: Cambridge University Press)
- [24] Berry M V and Upstill C 1980 Catastrophe optics: morphologies of caustics and their diffraction patterns *Prog. Opt.* **18** 257–346
- [25] Washimi H and Taniuti T 1966 Propagation of ion-acoustic solitary waves of small amplitude *Phys. Rev. Lett.* **17** 996–8
- [26] Ablowitz M J 2011 *Nonlinear Dispersive Waves: Asymptotic Analysis and Solitons* (Cambridge: Cambridge University Press)
- [27] Jeffreys H and Jeffreys B S 1956 *Methods of Mathematical Physics* (Cambridge: Cambridge University Press)
- [28] Godwin R P 1993 The hydraulic jump ('shocks' and viscous flow in the kitchen sink) *Am. J. Phys.* **61** 829–32
- [29] Rayleigh L 1914 On the theory of long waves and bores *Proc. R. Soc. A* **90** 324–8
- [30] Lamb S H 1932 *Hydrodynamics* (Cambridge: Cambridge University Press)